

Cash or Annuity?

Mathematics Focus:

Algebra II, Precalculus, Data Analysis/Probability

Mathematics Prerequisites:

Prior to using this lesson, students should be able to:

- ▶ Calculate simple interest
- ▶ Calculate annual compound interest
- ▶ Calculate interest compounded a finite number of times annually
- ▶ Calculate continuous interest

Lesson Objectives:

To apply prerequisite mathematics concepts and processes to consumer decision-making processes to:

- ▶ Explain that a dollar received today is worth more than a dollar received one year from today
- ▶ Explain the difference between the payments options of cash and annuity
- ▶ Explain interest compounding and its effect on the future value of money saved/invested for a period of time

Overview of Mathematics and Economics:

Jackpot winners of state lotteries may have the choice of receiving their winnings in the form of cash or an *annuity*. An annuity is a financial instrument that provides income at regular intervals over a specified time period. For example, New York's Lotto and California's SuperLotto offer an annuity with an annual payment for 26 years, and the multi-state Powerball lottery has a 25-year annuity option. Not many people will be lucky enough to win a lottery grand prize, but a great number of us will have a cash versus annuity option for receiving retirement distributions. These choices also exist for a number of other financial options, such as bonds and insurance accumulations. It is therefore important that students understand the economics of alternative payment options.

A fundamental principle of economics is that a dollar received today is worth more than a dollar received in one year's time. Economists use *present value* analysis to make comparisons between the value of payments received at different intervals. This lesson instructs students on the mathematics of financial payments options that they will encounter in

their lifetime. Among other things, students are asked to complete an activity in which they analyze the accumulations that will arise from making a commitment to saving a small amount of their weekly earnings. This involves calculating the future value of a stream of periodic payments.

Mathematics and Economics Terms:

- ▶ Annuity
- ▶ Compound interest
- ▶ Future value
- ▶ Interest
- ▶ Present value
- ▶ Principal

Materials:

- ▶ Calculator
- ▶ One copy of Visuals 11.1, 11.2, and 11.3 as well as Activities 11.1, 11.2, and 11.3 for each student
- ▶ Overhead transparencies of Visuals 11.1, 11.2, and 11.3

Estimated Time:

80 minutes

Procedures:

1. Warm-Up Activity: Distribute a copy of Visual 11.1 to each student and show a copy on the overhead. If you have not previously covered the calculation of compound interest in your class, the first four formulas will allow you to do so. If this has already been covered, ask students to identify the formulas associated with the particular form of compounding that is illustrated.

1) Total = $P(1 + nr)$

◆ Answer: This is the total value of an investment, P , after n years at interest rate r (when r is expressed as a decimal) when the interest is not compounded.

2) Total = $P(1 + r/k)^{nk}$

◆ Answer: This is the total value of an investment, P , after n years at interest rate r (when r is expressed as a decimal) when the interest is compounded k times annually. The value of nk is the total number of compoundings during the duration of the investment.

3) Total = $P(1 + r)^n$

◆ Answer: This is the total value of an investment, P , after n years at interest rate r (when r is expressed as a decimal) when the interest is compounded once annually.

$$4) \text{ Total} = P e^{rt}$$

Answer: This is the total value of an investment, P , after t years at interest rate r (when r is expressed as a decimal) when the interest is compounded continuously.

$$5) \text{ Total} = P (1 + r/n)^n = 4000(1 + .07)^{10} = \$7868.61$$

$$6) \text{ Total} = P(1 + r/n)^{nk} = 4000 (1 + .07/4)^{40} = \$8006.39$$

$$7) \text{ Total} = P e^{rt} = 4000e^{.07(10)} = \$8055.01$$

$$8) \text{ Total} = P(1 + nr) = 4000(1 + 10(.07)) = \$6800$$

2. Distribute a copy of Visual 11.2 to each student and project a copy on the overhead projector. Tell the students the following: As long as interest, the amount that a borrower pays a lender for the use of funds, can be earned, a dollar received today is more valuable than a dollar received a year from today. Have the students complete Visual 11.2 either individually or in groups.

The answers to the questions are listed below:

1. This item is descriptive only; no answer is needed here.
2. Original principal = (future value)/(1 + interest rate)
3. \$.926
4. Present value of the future dollar is \$.926
5. \$0.91

After going over the answers, ask how much the students would be willing to pay today for a certain amount of money received one year from today. Of course, the answer depends on what the interest rate is over the period. Other things equal, the higher the interest rate, the less they will be willing to pay today for the future dollar.

3. Explain to the students that an annuity is a sum of money payable yearly or at regular intervals. Often this choice is an alternative to a single cash payment now. Explain lottery option payments to students. Tell them that when states publicize the amount in the jackpot for an upcoming lottery drawing, they are almost always highlighting the annuity value of the grand prize. This is the total amount of payments that a winner can expect to receive over the next 25 or 26 years, not the amount of cash that the winner can expect to receive immediately. The cash equivalent of the annuity value is substantially less than the total reported jackpot amount. In fact, state lottery authorities typically suggest that the cash payout is about one-half of the publicized jackpot.

■ **COMMENT:** For purposes of this lesson, we are disregarding the taxes that must be paid on lottery winnings. To be sure, taxes reduce the amount of take home winnings, but since we are focusing on the mathematics of the cash versus annuity option, taxes are not included in the calculation.

4. Distribute a copy of Activity 11.1. Tell the students that the table in this activity is adapted from a table that California lottery authorities produced in June 2000. This table reports the payment amounts over the years for the winner of a hypothetical \$7 million jackpot of the Super Lotto game. The 26 payments sum up to \$7 million. Lottery authorities indicate that the cash equivalent option of this annuity would be about \$3.4 million. Ask students to use information given in this table in order to determine whether the authorities are correct. This involves calculating the present value of future annuity payments. At the time that this table was produced, the interest rate was approximately 6 percent (so that $r = 0.06$ in the present value calculation). Tell the students that this is the rate to be used.

Allow the students to work in groups to complete the table. You may want the students to round the present value to the nearest whole dollar. Then sum up the 26 present value calculations. You may wish to note that a present value calculation is not needed for the first payment (payment number 0), because this payment is received immediately. Students should be able to verify by their totals that the cash option of \$3.4 million is essentially equivalent to the future \$7 million value. The answers appear in the table below.

■ **COMMENT:** This is an excellent time to remind the students that they will confront this option repeatedly in their adult lives. This analysis is used when deciding on the type of pension you choose, how you choose to use funds from an insurance distribution, what type of bond you purchase, and even whether or not to pre-pay a home mortgage. This exercise is about present value analysis. It just turns out that the most visible example of this important concept is the state lottery.

Payment Number	Annual Payment	Present Value
0	\$175,000.00	\$175,000.00
1	\$189,000.00	\$178,301.89
2	\$196,000.00	\$174,439.30
3	\$203,000.00	\$170,442.71
4	\$210,000.00	\$166,339.67
5	\$217,000.00	\$162,155.02
6	\$224,000.00	\$157,911.16
7	\$231,000.00	\$153,628.19
8	\$238,000.00	\$149,324.14
9	\$245,000.00	\$145,015.12
10	\$252,000.00	\$140,715.48
11	\$259,000.00	\$136,437.97
12	\$266,000.00	\$132,193.85
13	\$273,000.00	\$127,993.05
14	\$280,000.00	\$123,844.27
15	\$287,000.00	\$119,755.07
16	\$294,000.00	\$115,732.01
17	\$301,000.00	\$111,780.69
18	\$308,000.00	\$107,905.89
19	\$315,000.00	\$104,111.60
20	\$322,000.00	\$100,401.12
21	\$329,000.00	\$96,777.13
22	\$336,000.00	\$93,241.71
23	\$343,000.00	\$89,796.46
24	\$350,000.00	\$86,442.49
25	\$357,000.00	\$83,180.51
	TOTAL	\$3,402,866.53

5. Tell the students that another calculation they will find useful in their life-time is related to the future value of the money they have saved. Give each student a copy of Visual 11.3. This is a seemingly complicated formula that is used to calculate the future value of savings when the same amount, an

annuity, has been saved over equal intervals during a pre-specified time period. This formula also requires knowledge of the interest rate over the investment period. While interest rates will certainly vary over time, this is still a useful approach to calculating the future value of savings. Note that the formula is written in the form of an annual rate of interest and an annual investment interval. If investments are made more frequently, then adjustments need to be made to the formula. Activity 11.2 deals with future value.

6. Distribute a copy of Activity 11.2 to each student. Suggest to the students that a one-pack-per-day smoker is likely to spend \$25 per week on cigarettes. Ask the students to imagine that instead of spending this money on cigarettes or something of equivalent cost, they saved \$25 per week for 30 years at a rate of 6 percent interest. What will the entire savings be worth at the end of 30 years? Students are asked to determine the future value of this annuity after 30 years.

◆ The answer is \$109,290.

■ **COMMENT:** This activity looks at the calculation of the future value of \$25 saved each week for 30 years. It draws on the importance of interest compounding. The \$25 weekly amount was intentionally chosen to illustrate the financial cost of smoking one pack of cigarettes per day for 30 years. This illustration does not take into account the fact that health insurance is more costly for smokers or that smokers have shorter life expectancies.

If you want to show students how this number is calculated, note that the formula they are using is the sum of a finite geometric series consisting of the following payments: $FV = \$25(1.001154)^{n-1} + \$25(1.001154)^{n-2} + \dots + \$25(1.001154)^3 + \$25(1.001154)^2 + \$25(1.001154) + \$25$. This series has 1560 terms and sums to \$109,290. Extension exercise 3 shows how this geometric series can be reduced to the formula on Visual 11.3.

8. Distribute a copy of Activity 11.3 to be completed for homework.

◆ **The answers are:**

1. .9524
2. .8333
3. .1486
4. .3769
5. \$20,020

Extension Activities:

1. Some state lottery sites on the Internet address the cash vs. annuity question. One such site (corresponding to the Lotto game in New York) can be found at www.nylottery.org/cashvsannuity.htm. Have your students find the payments schedule table and compute the cash value of

the annuity in the same way that they completed Activity 11.1. All of the information except the interest rate is given. The most realistic interest rate for you to use is probably the rate on 30-year U.S. Treasury Bonds. See the *Wall Street Journal* for a current 30-year bond rate.

2. Suppose a friend of yours buys a lottery ticket each day when he goes to the convenience store to buy coffee. Also suppose that he has been doing this for 20 years. The lottery ticket costs \$1 and the expected return on each ticket purchased is \$0.40. Assuming that your friend goes to the store five times each week, what is his expected loss each week?

◆ Answer: \$3. He expects to win $5 \cdot (\$0.40) = \2 each week, but the weekly cost of playing is \$5.

Over a 20-year period, what would your friend accumulate if he had saved his net loss each week in an account that earned 6 percent interest?

◆ Answer: $FV = \frac{\$3\{(1.001154)^{1040} - 1\}}{.001154} = \6027

3. Using the formula for the sum of a finite geometric series, $S_n = a_1 \frac{1 - r^n}{1 - r}$, develop the Future Value formula as given in Activity 11.2

from the series $FV = \$25(1.001154)^{n-1} + \$25(1.001154)^{n-2} + \dots + \$25(1.001154)^3 + \$25(1.001154)^2 + \$25(1.001154) + \$25$. This series has 1560 terms and sums to \$109,290.

◆ Answer: In the formula $S_n = a_1 \frac{1 - r^n}{1 - r}$, a_1 is the constant factor removed

from all the terms. The future value formula calls this PMT. The letter r denotes the ratio of one term to the term following it. That ratio is $\frac{1.001154^{n-1}}{1.001154^{n-2}} = 1.001154^{n-1-n+2} = 1.001154$.

If we let $S_n = \sum_{k=0}^{n-1} 25(1.001154)^k = 25 + 25(1.001154) + 25(1.001154)^2 +$

$25(1.001154)^3 + \dots + 25(1.001154)^k$ and

$rS_n = 1.001154 \sum_{k=0}^{n-1} 25(1.001154)^k = 25(1.001154) + 25(1.001154)^2 +$

$25(1.001154)^3 + 25(1.001154)^4 + \dots + 25(1.001154)^{k+1}$

then $S_n - rS_n = 25 - 25(1.001154)^n$

and $(1 - r)S_n = 25[(1 - 1.001154)^n]$

and $S_n = \frac{25(1 - 1.001154)^n}{1 - 1.001154} = 25 \left[\frac{1 - 1.001154^n}{-.001154} \right]$

$= 25 \left[\frac{1.001154^n - 1}{.001154} \right] = \text{PMT} \left[\frac{(1 + r)^n - 1}{r} \right]$

VISUAL 11.1 ▲ Warm-Up

For Questions 1 – 4 indicate whether the formula is associated with continuous compounding, annual compounding, no compounding, or multiple intra-year compounding, and explain the purpose of the formula.

1. Total = $P(1 + nr)$

2. Total = $P(1 + \frac{r}{k})^{nk}$

3. Total = $P(1 + r)^n$

4. Total = $P e^{rt}$

Match and solve each interest calculation problem, with the corresponding formula in 1-4.

5. Suppose \$4000 is invested at 7 percent interest and the investment is compounded annually. Find the investment value after 10 years.

6. Suppose \$4000 is invested at 7 percent interest and the investment is compounded quarterly. Find the investment value after 10 years.

7. Suppose \$4000 is invested at 7 percent interest and the investment is compounded continuously. Find the investment value after 10 years.

8. Suppose \$4000 is invested in an account with 7 percent interest for 10 years, but the interest is not compounded. Find the value of the investment after 10 years.

VISUAL 11.2 ▲ The Time Value of Money: Computing Present Value

A dollar received today is more valuable than a dollar received one year from today.

1. The formula for the future value of an original principal after one year with an interest rate, r , is

$$\text{Future value} = \text{original principal} (1 + r)$$

2. Solve this equation for original principal.

3. What would the original principal be if the interest rate was 8 percent and you wanted to end up with a dollar one year from now?

4. The question above is the same as “How much would you be willing to pay today for the future payment of \$1 received one year from today?”

Let the original principal be referred to as present value.

Present Value of payment received one year from today = $\frac{\text{Future Value}}{(1 + r)^1}$

In symbols: $PV = \frac{FV}{(1 + r)^1}$

where FV is the future value and r is the interest rate expressed in decimal form.

5. Calculate the Present Value of the future payment of \$1, one year from today when the interest rate $r = 10\%$.

A general Present Value calculation for payment n years from today:

$$PV = \frac{FV}{(1 + r)^n}$$

VISUAL 11.3 ▲ The Future Value of an Annuity

PMT represents Constant Annual Payment

r represents interest rate for each payment interval in decimal form

n represents the number of payment intervals in the future for which the computation is being made

$$\text{Future value} = \text{PMT} \frac{\{(1 + r)^n - 1\}}{r}$$

This formula can be used to calculate the future value of payments you make to an account for any future purpose.

ACTIVITY 11.1 ▲ The Lottery Jackpot: Cash or Annuity?

Lottery officials report that the size of the lottery jackpot for an upcoming drawing is \$7 million. But then they report that this is the annuity value of the jackpot. If you actually want cash, you will *only* get \$3.4 million (and that is before taxes). **Are the grand prizewinners being cheated if they take the cash option?** Fortunately, your mathematics and economics background allows you to answer this question. You know that if the present value of the future annuity payments sum up to \$3.4 million, then the cash option is equivalent to the annuity. Your job is to determine whether or not the \$3.4 million is the correct amount for the cash option.

In your analysis, assume that the interest rate, r , equals 6 percent. To complete this exercise, you must determine the present value of each annual payment. Use the future value formula in Visual 11.2 and solve for present value. When you are done with this, total your present value calculations. (Note that the first two calculations are already done for you.)

1. What is the total of the 26 present value amounts?
2. Is the lottery officials' cash option equivalent to the annuity amount?

In your calculations, round off the present value calculation to a whole dollar amount. Use your calculator to help you complete this activity.

ACTIVITY 11.1 (continued)

Payment Number	Annual Payment	Present Value
0	\$175,000.00	\$175,000.00
1	\$189,000.00	\$178,301.89
2	\$196,000.00	
3	\$203,000.00	
4	\$210,000.00	
5	\$217,000.00	
6	\$224,000.00	
7	\$231,000.00	
8	\$238,000.00	
9	\$245,000.00	
10	\$252,000.00	
11	\$259,000.00	
12	\$266,000.00	
13	\$273,000.00	
14	\$280,000.00	
15	\$287,000.00	
16	\$294,000.00	
17	\$301,000.00	
18	\$308,000.00	
19	\$315,000.00	
20	\$322,000.00	
21	\$329,000.00	
22	\$336,000.00	
23	\$343,000.00	
24	\$350,000.00	
25	\$357,000.00	
	TOTAL	

ACTIVITY 11.2 ▲ The \$100,000 Habit

Students often hear about the undesirability of smoking, but do you ever wonder about the long-run cost of buying cigarettes? Your teacher has already discussed how the future value of an annuity is calculated. Essentially this involves calculating how much an annual investment of equal amounts sums to over a pre-specified time period. This exercise asks you to apply this formula to weekly savings over a 30-year period. This savings amount, a modest \$25 per week, roughly corresponds to the amount that would be spent each week by a one-pack-per-day smoker. How much does the cost of one-pack-per-day add up to after 30 years?

The formula for the future value of an annuity needs to be modified because we are now considering a time interval that is shorter than one year. Because \$25 is to be saved each week, the interest rate must be adjusted to be a weekly rate of interest. The number of periods must also be adjusted to be the total number of weeks over 30 years. The appropriate formula for calculating the future value of an annuity when weekly contributions are made is:

$$FV = \text{Weekly Payment} \cdot \frac{\{(1 + \text{weekly interest rate})^n - 1\}}{\text{weekly interest rate}}$$

where n is the total number of weeks over the time period examined.

Suppose the interest rate is 6 percent expressed on an annual basis. This means that the weekly rate of interest is $0.06/52 = 0.001154$. The total number of weeks over the 30 year period is $n = 30 \cdot 52 = 1560$.

Suppose you save \$25 per week for 30 years at 6 percent per year. What will be the future value of this annuity after 30 years?

The title of this activity is **The \$100,000 Habit**. Do you agree?

ACTIVITY 11.3

1. Calculate the present value of \$1 received 1 year from today when the interest rate is 5 percent.
2. Calculate the present value of \$1 received 1 year from today assuming a 20 percent rate of interest.
3. What is the present value of a dollar received 20 years from now if the interest rate is 10 percent?
4. What is the present value of a dollar received 20 years from now if the interest rate is 5 percent?
5. What is the future value of \$20 deposited every week from 3rd grade through 12th grade if the interest rate is 12 percent annually but compounded weekly?